Euclid

Riley Moriss

October 21, 2025

I will probably not be too careful with providing the exact references. For this I have read [Euc56] which contains a detailed introduction and commentary on the text, [Euc08] which contains the greek text along side the english in one volume and [Mue81] which has a summary style as well as many philosophical and logical comments.

1 Preface

This is the first in a projected series of case studies in the history of mathematics I would like to do to better understand the history of the subject I am in. Here I will make some comments about this project in general.

There are lots of things about modern mathematics that I dont understand, in its informal capacity. How different fields interact, how they interact with the formalised axiomatic notions. How all these things inform one another. These are sort of a-temporal problems that I hope to gain perspective on by examining past texts.

On the other hand I am interested in the historical specifics of the development of concepts as well as notation. This is intertwined with the other questions too in a sticky way. For instance one might hear that Galois invented group theory, but we are taught that a group is a set with some binary operations, surely a contrivance arriving much later than Galois. So how are these historical facts (Galois work and the set theoretic notion of group) operating on one another. More broadly speaking math is often dubbed as a cumulative science, there are no backwards steps in math, but if the foundations are rewritten is this not a backwards step (we have to recheck all the translations)?

To these ends I would like to draw my own attention to the following things that might come through in historical works: ontological and axiomatic commitments, especially as related to modern notions that are later recast in set theory, as well as notation used and invented.

I am principally interested in these questions and so personal trivia etc about the broader (non-mathematical) history of the times will feature very little. I am however interested in the original thoughts of these authors and therefore insist on primary sources. Unfortunately im a modern mono-lingual man. I will have to rely on translations, and for ancient texts translations of works handed down. I would like to record some comments on these processes too when relevant.

doing modern math I have internalised a whole bunch of formalist philosphy. How can I strip this away? how was it done before, how is it really done now.

2 Provenance

Euclids elements were written sometime around 300 BC. The originals dont exist. The first English translation appeared in 1570. All modern translations are of the Heiberg edition of the Greek text. This edition of Euclid is a synthesis and reconstruction created in 1885 from several (not all complete) extant manuscripts (roughly eight) dating (again roughly) from the 10th century (one from 888 AD and several from 10th, 11th and 12th). Heiberg also had access to (very small) fragments from the 7th, 8th and 9th century, as well as literal scraps (a single definition as one of them) from before the first century AD (preserved by the eruption of Mt. Vesuvius). So what we read today is the smashing together of various medieval copies of older copies of Euclides elements.

Like any good lost piece of media its facsimiles have debates and traditions about which is the most exact. [Euc56]'s introduction §5 and §7 have a detailed discussion. It appears as though in the 4th century AD a man named Theon (of Alexandria) had produced an edition of Euclid, likely compiled and definitely edited from previous sources. Almost all the extant manuscripts directly state that they were copied from something based on the work of Theon. There is a single manuscript (Vatican MS.190) that does not mention him and along with other evidence it is believed that this text is of a more ancient genealogy (a copy of something that was not influenced by the work of Theon). Interestingly however the Vatican manuscript, although omitting many of Theons additions itself disagrees with the even more ancient scraps previously mentioned, where Theonic manuscripts agree! Thus Heath comments that "we cannot make out any family tree" (he really only says this about Theonic manuscripts, but loosely it holds). It is believed that even before Theon there were many edits to the text, and that roughly the text "was spoiled ... about the 3rd century", believing for instance that Sextus Empiricus owned an authentic copy.

Another yet separate tradition exists in the Arab world. Editions of Euclid were traded there in the 9th century and they have many differences with the European editions. In general Heiberg avoids the Arab tradition, believing for textual reasons that they are less accurate.

3 The Aristotelian Context

Aristotle (384 - 322) had many ideas about how science, argument and deductions should be done and presented. His influence, once explained, is clear in Euclid. Euclid is littered with Aristotelian terminology and structure. For instance, Proclus (a commentator on Euclid and teacher in Platoes academy under the reign of Plotinus) explains (Aristotle's ideas) that every theorem should be formatted according to an enunciation, a proof and a conclusion:

> The most essential (parts of a theorem) and those which are found in all are enunciation, proof and conclusion.

By this he means the following. Enunciation is a statement of the proposition to be proved. A proof is the demonstration itself and a conclusion is a statement of how the enunciation follows from the proof. It is evident that all of Euclids propositions follow closely this schema.

Another example is the very word "Elements". Quoting Heath now [Euc56]

> There are ... certain leading theorems, all-pervading and furnishing proofs of many properties. Such theorems are called by the name of elements.

This is simply applying the sort of atomic idea of elements, the things out of which all else is built, to the theory of geometry or mathematics.

More interesting to me is the thoughts of Aristotle on axioms, hypothesis, definitions and postulates. What follows is our summary of Aristotles position.

Domains of inquiry have their starting points, "it is impossible that there should be demonstrations of everything". Different demonstrative sciences take different things for granted, for instance geometry assumes things about lines and points, while arithmetic assumes things about numbers. The things which are necessary "for anyone to hold who is to learn anything whatever", are called axioms or "common principles". Note that is it the domain of the first philosopher to justify these axioms (although not completely).

After axioms there are further definitions. Definitions do not assert that the thing defined exist. This is the role of an axiom or a demonstration

> what is denoted by the first and those derived from them is assumed; but as regards their existence, this must be assumed for the principles but proved for the rest.

Thus is geometry we have to assume to notion of a line and point, we can define a triangle in terms of them, but we must then prove that such a thing exists.

Finally there are the postulates. Heath nicely summarises Aristotle discussion on postulates as follows

> Besides the common notions there are a few other things which I must assume without proof, but which differ from the comomn notions in that they are not self-evident. The learner may or may not be disposed to agree to them; but they must accept them at the outset and must be left to convince himself of the truth in course of the investigation.

Again it is clear that Euclid structures his elements in just such a way.

4 Axioms, Definitions and Postulates

Euclids formulation of geometry, though impressive, hardly lives up to the ideal Aristotle set out. Another comment is that arithmetic is dealt with in Euclid books (VII, IX).

4.1 Definitions

We start with the definitions, to at least know what the terms refered to later are. Euclid consitently adds to his list of definitions at the begining of each chapter, while the postulates and axioms are fixed in the first chapter. Lets have a sampling of the geometric definitions:

1. A point is that which has no part.
2. A line is breadthless length.
3. The extremities of a line are points.
4. A straight line is a line which lies evenly with the points on itself.
5. A surface is that which has length and breadth only.
6. The extremities of a surface are lines.
7. A plane surface is a surface which lies evenly with the straight lines on itself.
8. A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

9. And when the lines containing the angle are straight, the angle is called rectilinear.
10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.
11. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction

And the arithmetic defintions:

1. A unit is (that) according to which each existing (thing) is said (to be) one.
2. And a number (is) a multitude composed of units.
3. A number is part of a(nother) number, the lesser of the greater, when it measures the greater.
4. And the greater (number is) a multiple of the lesser when it is measured by the lesser.
5. An even number is one (which can be) divided in half.
6. And an odd number is one (which can)not (be) divided in half, or which differs from an even number by a unit.
7. A prime number is one (which is) measured by a unit alone.
8. And when two numbers multiplying one another make some (other number) then the (number so) created is called plane, and its sides (are) the numbers which multiply one another.
9. And when three numbers multiplying one another make some (other number) then the (number so) created is (called) solid, and its sides (are) the numbers which multiply one another.
10. A square number is an equal times an equal, or (a plane number) contained by two equal numbers.
11. And a cube (number) is an equal times an equal times an equal, or (a solid number) contained by three equal numbers.

Since these are just definitions it remains to be seen which ones are constructed and which ones are axioms. Many of these objects go unconstructed unfortunately, for example the plane surface. Points and straight lines are given by the postulates. One can see that there is no notation in the definitions only natural language, indeed through out there is only the labelling of lines by their end points (numbers are labelled in the same way).

One also notices that these definitions employ many undefinied concepts; consider the definition of parrallel lines where the concept of infinite is employed. Heath notest that this should not be read as employing a concept of *actual* infinity, but more like *potential* infinity, i.e. more like indefinitely. It is far from reducable to just the concept of point and line.

If you read the definitions and axioms you are justified in saying, what does any of this mean. Heath has a protracted commentary on this section. One can see that the debates, exegesis, and commentaries are plentiful. According to Heath one (modern) commentator could "make nothing" of the definition of the straight line. Another thing to notice is that for instance "defintion" 3 is not a definition at all. Heath explains that this double definition of a point is Euclides playing both sides in an ancient debate, is the point a monad with position or the extremity of a line.

Euclid uses the term "measure"; a length measures another if we could take the shorter of the two and using it as a ruler measure the longer of the two as an integer multiple of the first. So a magnitude measures another if one is a multiple of the other, Euclid says just this but it wasnt so clear.

4.2 Common Notions

These are the axioms of Euclid

1. Things equal to the same thing are also equal to one another.
2. And if equal things are added to equal things then the wholes are equal.
3. And if equal things are subtracted from equal things then the remainders are equal.
4. And things coinciding with one another are equal to one another.
5. And the whole [is] greater than the part

Some of these (2,3) are arithmetic in flavour, while others are more geometric (4, 5). The axioms here have been subject to debate over the millennia, are they really sufficient, what should be added, what can be removed. To my eyes, perhaps too modern, it is fruitless to worry about this in such vague terms.

Euclid is criticised for not including the *uniqueness* of these postulated lines. Apparently this property is used.

Postulate 4 can be thought of as the homogenaity of space. It also shows the dualism between an angle as a relation between lines and as the *magnitude* of the inclination.

4.3 Postulates

We once again quote the postulates

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

The fifth is the infamous parallel postulate, which for some reason people beleived should be a theorem. Euclid can be praised for his insight that in fact it does need to be assumed.

5 Remarks

The first proposition is

> To construct an equilateral triangle on a given finite straight line.

This then uses the postulates and common notions to show that triangles and in particular equalateral triangles (defined) exist. Proposition 5 on the other hand is a more traditional proposition

> For Isosceles triangles, the angles at the base are equal to one another...

In proposition 9 we have Euclid using (an unmentioned) an axiom of choice, saying

> Let the point D have been taken at random on the line AB.

There were two words in Greek used for point. The older word used by Aristotle roughly means a "puncture", whilst later writers such as Euclid used a word that literally means "a mark" (as in the mark on a page).

Heaths book is itself an interesting peice of history as he was a contemporary of Hilbert and Cantor and discusses their contributions to geometry and history.

Magnitudes in Euclid have a somewhat dual meaning, being the actual line that represents a length and the length said line. There was not notation for exact ratios like $10/7$ or $\sqrt{2}$.

There is an interesting discussion by Heath as to the notion of when two ratios are the same (not necissarily ratios of rational numbers). Euclids definition is *very similar* to the definition given by Dedekind (his cuts). I leave the details there.

Some further comments made here on the myth of the crisis of irrationals. The crux of the issue is that the triangle with a $\sqrt{2}$ side length to the greeks *was not a number*, that is geometric constructions did not create numbers. Lengths or magnitudes were not conflated with numbers, those things generated from unity. So even through thier theories were similar (see below) the two were seemingly seperate. Other links discussing this.

The greeks did have their notation for whole numbers, the numerals, in base 10, but without a 0.

5.1 Magnitude and Number

There is a conspicously confusing aspect of Euclid that seems to plague all modern readers. This is the seperation of Euclid into geometry and arithmetic, or the two chapters V and VII respectively. "Euclid develops the subject of arithmetic in almost complete isolation from the remainder of the elements". Mueller argues that "magnitudes are geometric objects only and do not include numbers" whilst Heath points out the Euclid could "hardly have failed to notice the similarity" between the two theories.

This is Muellers interpreation. "There are indefinitely many units a finite selection from which composes a number". The definitions of the numbers like 2 etc are taken for granted, as well as the property that for instance $1 + 1 = 2$. "If one makes allowance for the difference between numbers and geometric objects and hence for the somewhat different sense attaching to such notions as addition and quality, his arithmetic assumptions are basically generalisation of his geometric ones". So even though Euclid invites geometric intuition through his terminology and diagrams the actual logic is not reliant on his geometric theory. Indeed it seems as though arthmetic could be though of as geometry without the postulates.

On the other hand ratios are relations between geometric objects. In this way they do not have a natural ordering and so even though they share some resemblance of numbers, in that a magnitude can be considered as a unit, they are not a *number system* in the same way. "Diophantus was the first Greek mathematician who frankly recognized fractions as numbers." "Euclids failure to establish a correlation between his two treatments of proportionality ... is probably the greatest foundational flaw in the elements". This is made more confusing by the fact that one can *in some sense* perform "arithmetic" operations on magnitudes, such as addition and multiplication *sometimes*, discussed in more detail here.

Heath has this to add Heath conjectures that he simply wrote down the theory as it came to him, with two historical traditions.

> It is a remarkable fact that the theory of proportions is twice treated in Euclid, in Book v. with reference to magnitudes in general, and in Book VII. with reference to the particular case of numbers. The latter exposition referring only to commensurables may be taken to represent fairly the theory of proportions at the stage which it had reached before the great extension of it made by Eudoxus (pre-Platonic thinker).
>
> Yet Euclid says nothing to connect the two theories of proportion even when he comes in x. 5 to a proportion two terms of which are magnitudes and two are numbers (" Commensurable magnitudes have to one another the ratio which a number has to a number"

To see how the theory of ratios was actually elaborated (not just contrasting it with the theory of arithmetic) Heath has some discussion of this in §5. A question might be if I construct a line using my ruler and compass from the unit then in what proportion does it stand to the original unit, that is how much do I need to multiply it by in order to get a multiple of the original unit. This should be thought of as replacing fractions by

$$\frac{a}{b} = \frac{c}{d} \iff ad = bc$$

Heath gives a nice in depth illustration of this borrowed from De Morgan. This sort of construction quickly leads to building a right angle between two units and asking what ratio the hypotenuse stands to the edges. Famously this is "incommensurable" that is it is not a ratio, there is always an error.

6 Examples

- The platonic solids are constructed in book 13.
- The infinitude of the primes is Book IX, Proposition 20.
- X.3 is the Euclidean algorithm.
- Book X, proposition 2, gave nontermination of the Euclidean algorithm as a criterion for irrationality.
- proposition 5 of Book XIII immediately implies periodicity, and hence nontermination, of the Euclidean algorithm on the pair $\left(\frac{1+5}{\sqrt{2}}, 1\right)$
- In Books I. and II working by the representation of products of two quantities as rectangles, enables us to solve some particular quadratic equations.
- addition and subtraction are described in Book I, Propositions 2 and 3. The addition of polygonal regions is treated in Book I beginning in the proof of Proposition 35: "Parallelograms which are on the same base and in the same parallels equal one another." The discussion continues through the proof of the Pythagorean Theorem. As a matter of fact, Euclid's proof of the Pythagorean Theorem is itself an explicit procedure for slicing up two square regions and rearranging the parts to make a third square region which is their sum.

References

[Euc56] Euclides. The thirteen books of Euclid's Elements. Volume 1: Introduction and books I, II. Dover Publications, New York, second edition revised with additions edition, 1956. Num Pages: 432.

[Euc08] Euclides. *Euclid's elements of geometry: the Greek text of J.L. Heiberg (1883 - 1885): from Euclidis Elementa, edidit et Latine interpretatus est I.L. Heiberg, in aedibus B.G. Teubneri, 1883-1885*. s.n, s.l, revised and corrected edition, 2008.

[Mue81] Ian Mueller. *Philosophy of mathematics and deductive structure in Euclid's Elements*. Cambridge, Mass. : MIT Press, 1981.